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On the Changing Size Spectrum
of Particle Clouds
Undergoing Evaporation, Combustion,
or Acceleration

by Ascher H. Shapiro and Alve J. Erickson

Aerothermopressor Project

ON THE CHANGING SIZE SPECTRUM OF PARTICLE CLOUDS

UNDERGOING EVAPORATION, COMBUSTION, OR ACCELERATION*

by

Ascher H. Shapiro

Professor of Mechanical Engineering
Massachusetts Institute of Technology
Cambridge, Mass.,U.S.A.

(presently Visiting Professor of Applied Thermodynamics,
Engineering Laboratory, Cambridge University, Cambridge, England)

and

Alve J. Erickson

Assistant Professor of Mechanical Engineering
Massachusetts Institute of Technology
Cambridge, Mass.,U.S.A.

1. ABSTRACT

A theoretical treatment is given showing how the size distribution of a cloud of particles changes as the result of evaporation, combustion, or acceleration. The general differential equation governing the concentration of particles as a function of size, position and time is formulated for one-dimensional duct-type flows. Solutions to the differential equation are then obtained for a number of special problems of interest to evaporation and combustion.

When molecular transfers control, the equivalent mean diameter for evaporation or combustion of drops is found to be approximately constant with time.

This suggests that the conventional model of a constant number of uniform drops of varying size be replaced by a new model having a varying number of uniform drops of constant size. The new model predicts a lower rate of evaporation or combustion than the conventional model.

2. INTRODUCTION

The Problem

Many industrial processes require that a cloud of solid particles or liquid droplets interact with the gaseous (or, sometimes, liquid) phase in which they are dispersed. In certain of these processes, there is a spectrum of particle sizes, and, moreover, the particles change in size by reason of the interaction. Examples include: (a) evaporation of a cloud of liquid droplets; (b) growth of a liquid cloud by condensation; (c) combustion of either solid fuel particles or liquid fuel droplets. Normally the rate of growth (taken of course in an algebraic sense) of each particle will depend, among other things, on the diameter of the particle itself. Under such conditions, the shape of the particle spectrum will change as time proceeds, and this naturally introduces considerable difficulty into the analysis of the problem. Even when there is no change in particle size, the spectral distribution of sizes may change; for example, if the cloud is accelerated, different sizes of particles will accelerate at different rates, and the differences in particle speeds will alter the relative concentrations per unit volume of the several particle sizes.

Object and Plan of Investigation

It is the purpose of this paper to develop some understanding of how the existence of a spectrum of particle sizes, coupled with a size-dependent

growth rate, influences the processes described above.

First of all, we shall investigate in what way a given initial spectrum changes shape. This requires formulation and solution of the governing differential equation.

Secondly, we shall look into the question of whether processes of the type considered can indeed be treated by means of a simple model comprising a·cloud of particles of uniform "equivalent"size.

Most of the examples will refer to evaporation or combustion, with molecular transfer rates controlling. However, it is to be understood that the concepts and methods are equally applicable to other processes.

Previous Work

In virtually all theoretical investigations of evaporation, combustion, etc., the actual cloud of particles is replaced by a simple model comprising a constant number of drops of uniform but changing size.

The actual size distribution in the combustion of pulverized coal was considered by Hottel and Stewart [1] , but their treatment of the problem, being essentially numerical, was rather cumbersome and not adopted to general use.

Probert [2] treated the size spectrum during combustion by accounting for current drops within a certain size interval as the remains of larger drops existing at the beginning of combustion. Although not specifically stated by Probert, one interesting result obtainable from the calculations was the finding that the mean size of all droplets present in the combustion chamber in the steady state may, depending on circumstances, be either greater or smaller than the mean size of the injected droplets.

In the present paper, the governing equation of the particle spectrum is formulated in differential rather than integral form. Fortunately the equation is such that the general form of the solution for some cases may be found. The resulting analysis is therefore not only simple and straightforward, but applicable to a broad variety of practical problems.

3. NOMENCLATURE

A	cross-sectional area of duct
c_1, c_2, c_3	constants of integration
C_p	specific heat at constant pressure of surrounding medium
$\mathcal{D}$	diameter of particle
$\mathcal{D}^*$	constant reference diameter equal to $\mathcal{D}$ at the point $\left. \partial G/\partial \mathcal{D} \right]_{t=0} = 0$
$\mathbb{D}$	dimensionless diameter $\mathcal{D}/\mathcal{D}^*$
D/Dt	signifies "substantial" differentiation with respect to time, i.e., while following a particle of fixed identity
E	a constant, $\mathcal{L} \rho V_0 / m_g C_p \tau_0$
$f_1(\mathcal{D})$	see Eq. (13a)
$f_2(t)$	see Eq. (13a)
G	$\partial n/\partial \mathcal{D}$ number of particles of size $\mathcal{D}$ per unit volume of space and per unit interval of diameter
$\mathbb{G}$	dimensionless form of G, $\quad \mathbb{G} \equiv \mathcal{D}^* G/n_0$
h	coefficient of heat transfer; coefficient of mass transfer
H	signifies an arbitrary function of the indicated argument
J	a function of diameter, $\quad - \int d\mathcal{D}/f_1(\mathcal{D})$
K	constant in molecular growth rate law, $R = - K/\mathcal{D}$
K_1	constant in growth-rate law for radiant heat transfer, $R = K_1$

K_2 constant in Stokes' Law for terminal velocity, $\mu = K_2 \mathcal{D}^2$

$\mathcal{L}$ latent heat per unit mass of evaporating substance

m_g mass of gas per unit volume of space

n total number of particles per unit volume at time t

∂n number of particles per unit volume lying within the diameter interval from $\mathcal{D}$ to $\mathcal{D} + \partial\mathcal{D}$

N see Eq. (25b)

q rate of evaporation or combustion

R growth rate of particle, $D\mathcal{D}/Dt$

s longitudinal distance along duct

S see Eq. (25b)

t time

T temperature of particle

T_g temperature of surrounding medium

μ velocity of particle

V total volume of particles in unit volume of space

x mass fraction of particle cloud which has disappeared

α see Eq. (3)

β see Eq. (3)

γ exponent in $R \sim \mathcal{D}^{-\gamma}$

θ dimensionless time variable, $Kt/\mathcal{D}^{*2}$

θ_1 dimensionless time variable, $K_1 t/\mathcal{D}^{*}$

λ thermal conductivity; mass diffusivity

ρ mass density of particles

σ transformed distance for observer moving with cloud, $\sigma = s - \mu t$

τ temperature difference, $\tau \equiv T_g - T$

$(\overline{})$ signifies quantities pertaining to appropriate mean size for same total mass and same rate of change of mass

$()_\mu$ signifies quantities for conventional model having fixed number of particles of uniform but varying size, with correct initial mass and correct initial rate of change of mass

$()_o$ signifies quantities at $t = 0$

4. BASIC CONCEPTS AND DEFINITIONS

The Particle-Size Spectrum

Assuming that the actual histogram of the size distribution of particles may for purposes of analysis be replaced by a continuous curve, we define the particle concentration (or spectrum ordinate), G , as $G \equiv dn/d\mathcal{D}$ where dn is the number of particles per unit volume of space lying within the infinitesimal range of diameters between $\mathcal{D}$ and $\mathcal{D} + d\mathcal{D}$. Then a graph of G versus $\mathcal{D}$ (Fig. 1a) illustrates the size distribution in the cloud. The area under a narrow vertical strip, $G d\mathcal{D}$, represents dn , the number of particles per unit volume having diameters within the size range of the strip.

Dimensionless Representation. When the shape of the spectrum, rather than the actual values, is of interest, it is convenient to employ the dimensionless coordinates, $\mathcal{D}^* G/n_o \equiv \mathbb{G}$ and $\mathcal{D}/\mathcal{D}^* \equiv \mathbb{D}$, where $\mathcal{D}^*$ is any convenient but fixed value of $\mathcal{D}$ (for example, the value of $\mathcal{D}$ corresponding to the initial maximum value of G). These normalized coordinates are shown in brackets on Fig. 1a.

Total Number of Particles per Unit Volume. To find the total number of particles per unit volume of space, we take the total area under the

curve of Fig. 1a, inasmuch as

$$n = \int dn = \int_0^\infty G \, d\mathcal{D} = \int_0^\infty \mathcal{D}^* G \, d(\mathcal{D}/\mathcal{D}^*) = n_0 \int_0^\infty G \, d\mathbb{D} \qquad (1)$$

Total Volume of Particles. Similarly, the total volume of particles per unit volume of space, is given by

$$V = \frac{\pi}{6} \int_0^\infty \mathcal{D}^3 \, dn = \frac{\pi}{6} \int_0^\infty \mathcal{D}^3 G \, d\mathcal{D} = \frac{\pi}{6} \mathcal{D}^{*3} n_0 \int_0^\infty \mathbb{D}^3 G \, d\mathbb{D} \qquad (2)$$

and is therefore found by integrating under the spectrum curve after the ordinates of the latter have been weighted by the factor $\mathbb{D}^3$.

The Growth Rate

The most important feature of the present analysis is the changing size of the particles. This is described by the growth rate,

$$R \equiv D\mathcal{D}/Dt$$

where the operator D/Dt signifies "substantial" differentiation, i.e., R is the rate of increase of particle diameter for a particle of fixed identity. For evaporation or combustion, R would have the negative values.

The value of R depends on the type of process (evaporation, combustion, etc.); on the physical properties of the particles and surrounding medium (velocity, density, viscosity, thermal conductivity, temperature difference, etc.); and, most important for our present purpose, on the diameter $\mathcal{D}$ of the particular particle concerned.

Mean Particle Size

One may define any number of mean sizes [3] through the general

formula

$$\overline{\mathcal{D}}_{\alpha/\beta} \equiv \left[\frac{\int_0^\infty \mathcal{D}^\alpha G \, d\mathcal{D}}{\int_0^\infty \mathcal{D}^\beta G \, d\mathcal{D}} \right]^{\frac{1}{\alpha-\beta}} \tag{3}$$

where α and β are arbitrary numbers, and $\overline{\mathcal{D}}_{\alpha/\beta}$ is the corresponding mean diameter. For example, with $\alpha = 3$ and $\beta = 2$, the value of $\overline{\mathcal{D}}_{3/2}$ thus obtained is the so-called volume-surface mean diameter, i.e., the particle diameter whose ratio of volume to surface is the same as that of the actual particle cloud.

Equivalent Mean Size for Evaporation or Combustion. More pragmatically, we restrict the definition of mean size to one which is of some use. Our real aim is to treat the actual particle cloud as though it were composed of uniformly-sized particles. For the processes under consideration, it is evident that the actual cloud and the model cloud must agree in two respects: they must, instantaneously, have (i) the same total mass and (ii) the same rate of change of total mass. Using bars to denote quantities referring to the model cloud of uniform drops, these requirements may be expressed as

$$\int_0^\infty \mathcal{D}^3 G \, d\mathcal{D} \equiv \overline{n}\, \overline{\mathcal{D}}^3$$

$$\int_0^\infty \mathcal{D}^2 R G \, d\mathcal{D} \equiv \overline{n}\, \overline{\mathcal{D}}^2 \overline{R}$$

These may be solved simultaneously for $\overline{\mathcal{D}}$ and $\overline{n}$ to give

$$\frac{\overline{\mathcal{D}}}{\overline{R}} = \frac{\int_0^\infty \mathcal{D}^3 G \, d\mathcal{D}}{\int_0^\infty \mathcal{D}^2 G R \, d\mathcal{D}} \tag{4}$$

$$\overline{n} = \frac{1}{\overline{R}^3} \frac{\left[\int_0^\infty \mathcal{D}^2 R G \, d\mathcal{D}\right]^3}{\left[\int_0^\infty \mathcal{D}^3 G \, d\mathcal{D}\right]^2} \tag{5}$$

At each instant, the model cloud containing $\bar{n}$ particles of uniform diameter $\bar{\mathcal{D}}$ will have the same mass and the same rate of evaporation or combustion as the actual cloud. Here it is important to note, however, that $\bar{n}$ is not equal to the number of particles in the actual cloud; nor does it remain constant with the passage of time.

Examples of Practically-Significant Mean Sizes. Often the dependence of R on $\mathcal{D}$ may be approximated by $R \sim \mathcal{D}^{-\gamma}$. The constant γ is zero when radiant heat transfer is controlling; it is equal to unity when molecular transfers control; and it is equal approximately to 0.2 when turbulent transfers control.

From Eq. (4), the mean size is then

$$\bar{\mathcal{D}}^{(1+\gamma)} = \frac{\int_0^\infty \mathcal{D}^3 G \, d\mathcal{D}}{\int_0^\infty \mathcal{D}^{2-\gamma} G \, d\mathcal{D}} \tag{4a}$$

The two limiting values of γ may briefly be anticipated here:

(1) If R is independent of $\mathcal{D}$, as would be the case when the process depends on radiant heat transfer, then $\gamma = 0$, and the appropriate mean is the volume-surface mean,

$$\bar{\mathcal{D}}_{\gamma=0} = \bar{\mathcal{D}}_{3/2} = \frac{\int_0^\infty \mathcal{D}^3 G \, d\mathcal{D}}{\int_0^\infty \mathcal{D}^2 G \, d\mathcal{D}} \tag{4b}$$

(2) If the process depends primarily on molecular transfers (i.e., the relative Reynolds number is very small), then $\gamma = 1$ [4], and the appropriate mean size is the 3/1 mean,

$$\bar{\mathcal{D}}_{\gamma=1}^2 = \bar{\mathcal{D}}_{3/1}^2 = \frac{\int_0^\infty \mathcal{D}^3 G \, d\mathcal{D}}{\int_0^\infty \mathcal{D} G \, d\mathcal{D}} \tag{4c}$$

5. DIFFERENTIAL EQUATION GOVERNING
HISTORY OF PARTICLE SPECTRUM

Description of Model of Process

We consider here the flow of a discrete particle cloud in a duct
or stream tube (the latter referring to the particle flow) of variable
cross-sectional area, A. The flow is considered one-dimensional to the
extent that, over each cross-sectional area, the particle concentration, G,
the particle speed, u, and the particle growth rate, R, are all uni-
form for each particle diameter, D. G, u and R are all considered
to depend on particle size, longitudinal location, and time, e.g., $u = u(D, s, t)$

The Growth Flux

A necessary preliminary to the differential equation is the deriva-
tion of an expression which indicates the flux rate at which, as a con-
sequence of growth, particles cross from sizes smaller than D to sizes
larger than D.

Consider all the particles smaller than D in a unit volume of
space at time t. During the time interval dt, all the particles
will have grown at their appropriate growth rates. The increase of diameter
of those particles originally of size D is $dD = R dt$, and the number
of particles grown larger than D is therefore $G R dt$ (see Fig.1b).
Dividing this expression by dt, we obtain an expression for the growth
flux:

$$\left\{ \begin{array}{l} \text{Number of particles per unit time and} \\ \text{per unit volume becoming larger than } D \end{array} \right\} = G R \qquad (5)$$

Governing Differential Equation

We shall make a numerical accounting of only those droplets lying within the size range σ to $\sigma + d\sigma$ in the control space of Fig. 2. At the location s , let the area be A and let the properties corresponding to σ be denoted by μ , G , etc.; at the exit location $s + ds$, each will at the same instant be larger by differential increments $(dA/ds)\,ds$, $(\partial \mu / \partial s)\,ds$, $(\partial G / \partial s)\,ds$ etc.

The numerical accounting of droplets in the size range from σ to $\sigma + d\sigma$ requires a consideration of the convective fluxes into and out of the control space, of the growth fluxes into and out of the size interval, and of the rate of change of particle number within the control space and size interval.

Convective Fluxes. At location s , the number per unit volume of droplets in the size range σ to $\sigma + d\sigma$ is $G\,d\sigma$. Multiplying this by μA , i.e., by the volume of space swept out per unit time by droplets of this size, we get an expression for the convective flux entering and a corresponding one for the instantaneous flux leaving:

Rate of entry into control space $= \mu A G\, d\sigma$ 　　(6a)

Rate of exit from control space $= \mu A G\, d\sigma + \dfrac{\partial}{\partial s}\left(\mu A G d\sigma\right) ds$ 　(6b)

Growth Fluxes. Particles smaller than σ grow into the size range under consideration, and may, for our present purpose, be said to be created. Likewise, particles larger than $\sigma + d\sigma$ grow out of the size range and may be said to be destroyed. Taking note of Eq. (5), we may accordingly write

Birth rate within control space $= G R A\, ds$ 　　(7a)

Death rate within control space $= G R A\, ds + \dfrac{\partial}{\partial \sigma}\left(G R A\, ds\right) d\sigma$ 　(7b)

Numerical Accounting. The instantaneous number of particles within the considered size range and lying inside the control space is equal to $(G\,d\varpi)(A\,ds)$. By setting the time rate of change of this quantity equal to the net rate of entry into the control space [Eq. (6a) minus Eq. (6b)], plus the net rate of creation within the control space [Eq. (7a) minus (7b)], and simplifying, we obtain the differential equation governing the time and space histories of the particle spectrum:

$$A \frac{\partial G}{\partial t} = - \frac{\partial}{\partial s}(\mu A G) - A \frac{\partial}{\partial \varpi}(G R) \tag{8a}$$

or

$$\frac{1}{G}\left[\frac{\partial G}{\partial t} + \mu \frac{\partial G}{\partial s} + R \frac{\partial G}{\partial \varpi}\right] = -\frac{\partial \mu}{\partial s} - \frac{\mu}{A}\frac{dA}{ds} - \frac{\partial R}{\partial \varpi} \tag{8b}$$

Now, since $G = G(t, s, \varpi)$, we may write

$$\frac{dG}{dt} = \frac{\partial G}{\partial t}\frac{dt}{dt} + \frac{\partial G}{\partial s}\frac{ds}{dt} + \frac{\partial G}{\partial \varpi}\frac{d\varpi}{dt}$$

Specializing this for a group of particles of fixed identity, for which $ds/dt = \mu$, and $d\varpi/dt \cdot R$, we obtain an expression for the rate of change of concentration of particles of fixed identity, identified by the substantial derivative notation, in the form

$$\frac{DG}{Dt} = \frac{\partial G}{\partial t} + \mu \frac{\partial G}{\partial s} + R \frac{\partial G}{\partial \varpi} = -G\left[\frac{\partial \mu}{\partial s} + \frac{\mu}{A}\frac{dA}{ds} + \frac{\partial R}{\partial \varpi}\right] \tag{8c}$$

where the second equality is found by reference to Eq. (8b).

Equation (8c) shows mathematically what might be deduced as well from purely physical considerations: that the spatial concentration of a

certain group of particles of fixed identity (a) is decreased when the particles accelerate; (b) is decreased when the particles pass to a section of greater cross-sectional area; and (c) is decreased when the growth rate increases algebraically with diameter.

General Form of Solution

To determine the time and space histories of the size spectrum, it is necessary to find the solution of Eq. (8), subject to the appropriate initial conditions of the problem.

Equation (8) is a linear partial differential equation of first order. The theory of such equations [5] shows that the general solution of this equation is of the form

$$H\left(\nu_1, \nu_2, \nu_3\right) = 0 \qquad (9a)$$

where $\nu_1(t, s, \mathcal{D}, G) = c_1$, $\nu_2(t, s, \mathcal{D}, G) = c_2$ and $\nu_3(t, s, \mathcal{D}, G) = c_3$ (in which c_1, c_2 and c_3 arbitrary constants of integration) are independent solutions of the associated ordinary differential equations

$$\frac{dt}{1} = \frac{ds}{\mu} = \frac{d\mathcal{D}}{R} = \frac{-dG}{G\frac{\partial \mu}{\partial s} + \frac{\mu G}{A}\frac{dA}{ds} + G\frac{\partial R}{\partial \mathcal{D}}} \qquad (9b)$$

In practice, the solutions of Eq. (9b) may be troublesome, and may indeed not be possible except in special cases where the variables may be readily separated.

We therefore proceed to the consideration of some special cases with the aim of discovering typical facts concerning the behavior particle spectra.

6. THE UNIFORMLY-MOVING CLOUD

WITH SIZE CHANGE

Specification of Model

Consider the case where all the particles move at the same constant speed in a duct of constant area. Such might occur when the particles were carried along in a fluid stream and all particles had settled down to the constant fluid velocity. The particle spectrum, however, changes as the result of evaporation, combustion, etc. Then we have the simplifications that $dA/ds = 0$, $\partial u/\partial s = 0$, and $\partial u/\partial \mathcal{D} = 0$

Simplified Differential Equation

With the foregoing assumptions, Eq. (8) now becomes

$$\frac{\partial G}{\partial t} + u \frac{\partial G}{\partial s} = -\frac{\partial}{\partial \mathcal{D}} (GR) \tag{10}$$

Transformation to Observer Moving with Cloud. If we consider an observer moving with the particle cloud at the speed of the stream, the stream and the imbedded particle cloud will all appear stationary. Therefore, it is convenient to define a distance coordinate

$$\sigma \equiv s - ut \tag{11}$$

which is the relation between distance s in the fixed reference frame and distance σ in the moving observer's reference frame. Applying the usual procedure for interchange of variables from the t, s, $\mathcal{D}$ system to the $t, \sigma, \mathcal{D}$ system, we transform Eq. (10) into

$$\left(\frac{\partial G}{\partial t} \right)_{\sigma, \mathcal{D}} = - \left[\frac{\partial}{\partial \mathcal{D}} (RG) \right]_{\sigma, t} \tag{12}$$

In fact, we may drop the subscript σ altogether if we remember that Eq. (12) is valid only when the observer moves with the cloud and that

the resulting equation applies to a group of particles of fixed identity.

Steady-Flow Case. It may be noted that Eq. (12) includes the special case in which the process is steady in the stationary reference frame (i.e., $\left[\partial G/\partial t\right]_{s,\sigma} = 0$), but of course non-steady in the reference frame of the moving observer.

Motionless Cloud in Stationary Medium. Also covered by Eq. (12) is another case of particular interest, that in which the particle cloud is motionless in a fixed space, the particles either growing or diminishing in size as time proceeds. The distance coordinate σ is then evidently irrelevent, and may be struck from Eq. (12); this may also be seen more directly by setting $\mu = 0$ in Eq. (10).

Growth Rate Considerations

Equations (9b) are indeed simplified, as we have seen, by dropping the terms $\partial \mu/\partial s$ and dA/ds. To obtain the solution even to simplified cases, however, we must know how the growth rate R depends on D and t. Now R depends on t to the extent that the physical properties of the fluid medium (such as the temperature, concentration, thermal conductivity, etc.), and of the particles, partly govern the rates of evaporation, combustion, etc., and these properties may all be functions of time. For example, the temperature difference forcing heat transfer depends on how much has already been evaporated or combusted.

Since the form of the R-t relationship is not known in advance, it must be found by simultaneously solving the spectrum equations and the equations governing the changes in properties of the fluid medium and of the particles. Consequently, the solution must involve such auxiliary relations as the energy equation, stoichiometric equations, heat transfer relationships, etc., inasmuch as these enter into the determination

of the rates of interaction between the cloud and the fluid medium as well as of the properties of the fluid medium.

Case for which $R = f_1(\mathcal{O}) f_2(t)$

Although the solution for G as a function of t and $\mathcal{O}$ cannot be found for the general case until the R-t relationship is known, we can avoid the simultaneous solution of the spectrum and fluid medium equations by assuming that the growth rate can be written as

$$\frac{D\mathcal{O}}{Dt} \equiv R = f_1(\mathcal{O}) f_2(t) \tag{13a}$$

wherein the function f_1 depends on $\mathcal{O}$ alone and f_2 on t alone.

With this assumption we are now able to first solve the spectrum equations, and then determine the complete relationships for the interaction between the particle cloud and the medium. The physical situation to which Eq. (13a) applies is the case of a uniformly moving cloud and medium in which (i) all particle properties are the same except the size, and (ii) the properties of the fluid medium appear the same to each particle. Many real problems are approximately of this type; see, for instance, the example given in Section 8.

General Form of Solution for Uniformly-Moving Cloud. Referring to Eq. (9b) again, the solution to the dt, ds equation is now

$$s - \mu t = \sigma = c_1$$

the dt, $d\mathcal{O}$ equation has the solution

$$J + \int_0^t f_2 \, dt = c_2$$

where $\qquad J(\mathcal{O}) \equiv -\int d\mathcal{O}/f_1 \tag{13b}$

and the $d\mathcal{O}$, dG equation integrates to

$$G f_1 = c_3$$

Therefore, from Eq. (9a), the complete solution to the problem must be of the form

$$G = \frac{1}{f_1} H\left[\sigma, \left(J + \int_0^t f_2\, dt\right)\right] \tag{14}$$

where H is an arbitrary function of the arguments indicated.

Method of Determining Form of Arbitrary Function. To determine the precise form of the function H in a particular case, we must have f_1 and f_2 expressed algebraically in terms of $\mathcal{O}$ and t, respectively, and the relationship between G, $\mathcal{O}$, and σ must be known at some time, say $t = 0$. Putting this latter relationship into the form indicated by Eq. (14), one obtains the solution at any other time merely through replacing J at $t=0$ by $(J + \int_0^t f_2\, dt\ \ )$ at any other time.

Graphical Interpretation of Solution. The solution represented by Eq. (14) has a simple graphical interpretation. Focussing attention on a particular part of the cloud, i.e., $\sigma =$ constant, we suppose that the spectral curve $G(\mathcal{O})$ is known at time $t = 0$. Let this function be plotted (see dashed curve in Fig. 3) in the form $G f_1$ versus J. At any later time t_1, the corresponding curve will be displaced without change of shape, as indicated by the solid curve of Fig. 3. According to Eq. (14), the value of $G f_1$ for a given value of J at time t_1 must be equal to the value of $G f_1$ at $t=0$ corresponding to a value of $J_{t=0} = J_{t_1} + \int_0^{t_1} f_2\, dt$. Hence the curve for time t_1 is simply displaced leftwards from that for $t=0$ precisely by the magnitude $\int_0^{t_1} f_2\, dt$.

Propagative Character of Solution. The foregoing features indicate a propagative behavior to the solution. For a given portion of the cloud, the product $G f_1$, is seen to be constant for combinations of $\mathcal{O}$ and t corresponding to a constant value of the function $(J + \int_0^t f_2\, dt\ \ )$.

By separating variables and integrating between the limits $t = 0$, $\mathcal{O} = \mathcal{O}_{t=0}$ and $t = t$, $\mathcal{O} = \mathcal{O}$, we find from Eq. (13a), for a particle

of fixed identity,

$$\overline{J_{\mathcal{D}}} + \int_{o}^{t} f_{2} \, dt = \overline{J}_{\mathcal{D}_{t=0}} \tag{13c}$$

where $\overline{J_{\mathcal{D}}}$ is the value of J at time t , and $\overline{J}_{\mathcal{D}_{t=0}}$ is the value of J <u>for the same particle</u> at $t=0$. In other words, particles of fixed identity have a $\mathcal{D}$, t history specified by a constant value of $J + \int_{o}^{t} f_{2} \, dt$, which, by Eq. (13c), in the value $\overline{J}_{\mathcal{D}_{t=0}}$ associated with the particle at time zero. Then Eq. (14) may be interpreted as meaning that the product $G f_{1}$ remains constant for particles of fixed identity.

Thus the propagative behavior noted above is associated with the fact that the initial concentration of particles in a certain size range controls the concentration of the same particles, but in successively different size ranges, at all later times.

<u>Alternative Derivation of Solution by Physical Considerations.</u> The propagative behavior may also be brought out by a derivation of Eq. (14) based on more physical considerations. Consider a group of particles initially ($t=0$) lying within a certain narrow size range, $d\mathcal{D}_{t=0}$. Since the <u>number</u> of such particles at time t is equal to the number at $t=0$, we may write

$$d n = G_{\mathcal{D}_{t=0}} \, d\mathcal{D}_{t=0} = G_{\mathcal{D}_{t}} \, d\mathcal{D}_{t}$$

where $d\mathcal{D}_{t=0}$ and $d\mathcal{D}_{t}$ are, respectively, the initial and final widths of the size interval, and $G_{\mathcal{D}_{t=0}}$ and $G_{\mathcal{D}_{t}}$ are the concentrations corresponding to particles of size $\mathcal{D}_{t=0}$ at time $t=0$ and of size $\mathcal{D}_{t}$ at time t , respectively. By the definition of J $\left[\text{Eq. (13b)} \right]$,

$$d\mathcal{D}_{t=0} = - f_{1}(\mathcal{D}_{t=0}) \, dJ_{\mathcal{D}_{t=0}} ; \quad d\mathcal{D}_{t} = - f_{1}(\mathcal{D}_{t}) \, dJ_{\mathcal{D}_{t}}$$

But, for particles of fixed identity, Eq. (13c) holds. Differentiating this at constant time, we have $d\overline{J_{\mathcal{D}}} = d\overline{J}_{\mathcal{D}_{t=0}}$. Now, assembling the foregoing equations, we get

$$\left(G f_{1} \right)_{\mathcal{D}_{t}} = \left(G f_{1} \right)_{\mathcal{D}_{t=0}}$$

which states that the product $G f_1$, associated with particles of size $\mathcal{D}$
at time t is equal to the product $G f_1$ associated with the size $\mathcal{D}_{t=0}$ be-
longing to the same particles at $t=0$. But we have already seen that the
identity of a certain group of particles is marked by a $\mathcal{D}, t$ relationship
given by Eq. (13c). Consequently, we may conclude that the product $G f_1$
is a function only of $\left(J + \int_0^t f_2 \, dt\right)$, which is exactly what is claimed by Eq.(14).

Case of Constant and Uniform Growth Rate

If the growth rate is the same, and constant, for all drops, the
integral J of Eq. (13) is simply $J = -\mathcal{D}/R$, and the general solution
to Eq. (15) is $G R = H(\mathcal{D} - Rt)$, which may be expressed alternatively as
$G = H_1(\mathcal{D} - Rt)$. This states exactly what the assumption of constant
growth rate for all particles implies, namely, that the distribution curve
of G vs. $\mathcal{D}$ marches across the graph with unchanged form, the displace-
ment at any instant being Rt.

A possible application of this result is to problems in which the
growth rate is controlled by radiant heat transfer. An energy balance
in that case shows that R is independent of particle size.

Method of Treating Formation of New Drops

When the growth rate is positive, as in condensation, attention must
be given to the fact that new drops may constantly be born. The type of
solution already discussed applies only to those drops already present
at time zero. Assuming that, at each instant, the birth rate of new
drops per unit volume of space is known, we may equate this to the in-
stantaneous value of $G R$ corresponding to $\mathcal{D} = 0$. This in turn yields $G_{\mathcal{D}=0}$
as a function of time. Consequently, in respect to the new drops, the
form of the arbitrary function in Eq. (14) may be determined from a knowledge

of its variation with t for the value of J corresponding to $\mathcal{D} = 0$, and thus the general solution for the spectral history of the new drops may be found.

7. EVAPORATION OR COMBUSTION OF A CLOUD IN A VERY LARGE MEDIUM

Description of Process

To illustrate more completely the method outlined, and also to obtain some practical results of value to an understanding of evaporation and combustion, we now take up the special case of a stationary cloud evaporating into or burning in a very large, stationary, gaseous medium. As indicated by the preceding section, the results are also applicable to each portion of fixed identity of a uniformly-moving cloud. In this section we shall assume that the mass of the medium, relative to that of the cloud, is so large that the process does not materially alter the properties of the medium.

With this assumption, the growth rate for any portion of the cloud (i.e., for fixed σ) will depend only on diameter, and we may at once set $f_2(t) = 1$. Then, remembering that R is now only a function of $\mathcal{D}$, Eq. (14) becomes

$$ G = \frac{1}{R} \, H \, (J + t) \tag{15} $$

where H is an arbitrary function of the indicated argument. Alternatively, for any fixed portion of the cloud, Eq. (15) states that the product $G R$ (which varies only with $\mathcal{D}$) depends only on the combined function of $\mathcal{D}$ and t denoted by $(J + t)$. As before, the precise form of this dependency may be found by expressing $G(\mathcal{D})$ at $t = 0$ in the form of Eq. (15), and then replacing $J(\mathcal{D})$ where it appears by $\left(J(\mathcal{D}) + t \right)$.

The Molecular Growth-Rate Law

After the particles have reached a quasi-steady temperature, both evaporation and combustion are controlled by the processes of heat

transfer and mass diffusion. If we assume no relative motion between the particles and the medium (or, more generally, that the Reynolds number based on the relative speed is small), dimensional considerations require that the coefficients of heat transfer and mass transfer follow the law $h\mathcal{D}/\lambda$ = constant, where h is, respectively, the coefficient of heat transfer or mass transfer, and λ is, respectively, the thermal conductivity or mass diffusivity [4].

Since the rate of the process, and, consequently, the rate of disappearance of mass from each particle, is proportional to the coefficient h, to the surface area, and to differences in temperature and concentration, we may write that $\mathcal{D}^2(D\mathcal{D}/Dt) \sim h\mathcal{D}^2$. Then noting that $h\mathcal{D}$ is constant, we find that

$$R \equiv D\mathcal{D}/Dt = -K/\mathcal{D} = f_1 \tag{16}$$

where K is a positive constant containing such quantities as pure numbers, the density and latent heat of the particles, the temperature difference, and the thermal conductivity and mass diffusivity of the medium. Thus we may evaluate

$$J = -\int \frac{d\mathcal{D}}{f_1} = \int \frac{\mathcal{D}\,d\mathcal{D}}{K} = \mathcal{D}^2/2K \tag{17}$$

Moreover, integration of Eq. (16) for a particle of fixed identity yields

$$\int_{\mathcal{D}_{t=0}}^{\mathcal{D}} \mathcal{D}\,d\mathcal{D} = -K\int_0^t dt \; ; \quad \mathcal{D}_{t=0}^2 - \mathcal{D}^2 = 2Kt \tag{18}$$

where it is to be understood that $\mathcal{D}_{t=0}$ is the diameter at $t=0$ of a particle having the diameter $\mathcal{D}$ at time t.

General Solution

According to Eq. (15), the general solution to the problem considered here is

$$G = \mathcal{D}\,H\left[\mathcal{D}^2 + 2Kt\right] \tag{19a}$$

Rearranging this in dimensionless form, we have

$$\mathbb{G} \equiv \varnothing^* G / n_o = (\varnothing/\varnothing^*) H\left[(\varnothing/\varnothing^*)^2 + 2\theta\right] = \mathbb{D} \, H\left[\mathbb{D}^2 + 2\theta\right] \qquad (19b)$$

where $\theta \equiv Kt/\varnothing^{*2}$. The advantage of the form shown in Eq. (19b) is that it removes the results from any particular scale of time, size, or K, and expresses them in their most general form in terms of the dimensionless time, θ .

Graphical Method of Solution

A simple and rapid method of graphical solution is suggested by Eq. (19b). At time $t = 0$, the size distribution is plotted in the form of $\mathbb{G}/\mathbb{D}$ versus $\mathbb{D}^2$. Then, for any other time $t = t_1$, the corresponding curve may be found merely by sliding the original curve leftwards by the amount $2\theta = 2Kt_1/\varnothing^{*2}$, as illustrated by Fig. 4. Then it is a simple matter to regraph the new curve for time t_1 in the conventional form of $\mathbb{G}$ versus $\mathbb{D}$ or G versus $\varnothing$.

Analytic Solution of Typical Case

These are several types of size distribution which may represent atomized sprays or milled solid particles [6] . All have the features that the curve of G versus $\varnothing$ starts from zero, increases with $\varnothing$ and then, after reaching a maximum, approaches zero again asymptotically.

Initial Size Distribution. We consider now a specific type of distribution at $t = 0$ that embodies the foregoing features and that allows us to draw significant conclusions by simple calculations. The distribution postulated at $t = 0$ is represented by

$$\varnothing^* G = n_o \left(\varnothing/\varnothing^*\right) e^{-\frac{1}{2}(\varnothing/\varnothing^*)^2} \quad \text{or} \quad \mathbb{G} = \mathbb{D} \, e^{-\mathbb{D}^2/2} \qquad (20a)$$

For very small values of θ this gives a linear distribution, while for large values the concentration approaches zero very rapidly. The constants in the equation are chosen so that the initially most populous particles (i.e., those of maximum G) are of size θ^* , and the total number of particles initially is n_o . For the process considered, the instantaneous appropriate mean size is given by Eq. (4c). When the latter is evaluated for the distribution of Eq. (20a), we get $\overline{D} \equiv \overline{\theta}/\theta^* = \sqrt{3}$, where it is understood that $\overline{D}$ means, for this case, $\overline{D}_{3/1}$.

Size Distribution at Any Time. Comparison of Eq. (20a) with Eq. (19b) shows that the form of H at $t=0$ is

$$H_{t=0} = e^{-D^2/2}$$

Consequently, at any other time t , the spectrum is given by

$$G = D \, e^{-\frac{1}{2}[D^2 + 2\theta]} = e^{-\theta} \, D \, e^{-\frac{1}{2} D^2} \tag{20b}$$

This remarkable result shows that the size distribution considered remains unchanged in form, the concentrations of particles of all sizes being reduced by the constant factor $e^{-\theta}$.

Variation of Equivalent Mean Size. Performing the integrations of Eq. (4c), we get the striking result that $\overline{D} = \sqrt{3}$ at any time. That is, the appropriate mean size does not change, even though all particles are becoming smaller. The explanation of this seeming paradox is that the small particles grow smaller more rapidly than the large ones, and ultimately disappear entirely; as time proceeds, therefore, a relatively greater proportion of large particles remains.

Variation of Total Number of Particles. From Eqs. (1) and (20b), we find that the total number of particles decreases exponentially with time, i.e.,

$$\frac{n}{n_o} = e^{-\theta} \tag{20c}$$

Variation of Fraction of Mass Disappeared. The fraction of the total volume of all particles lost, which is a measure of how much mass has disappeared through evaporation or combustion, is found from Eqs. (2) and (20b) as

$$\chi \equiv \quad \text{Fraction Disappeared} = 1 - \frac{V}{V_{t=0}} = 1 - e^{-\theta} \qquad (20d)$$

Variation of Rate of Evaporation or Combustion. The rate of evaporation or combustion is proportional to the rate of disappearance of total volume of all particles, i.e., $q \sim - dV/dt$. From Eq. (1) and (20d), therefore, we find that this rate also varies exponentially with time,

$$q/q_o = e^{-\theta} \qquad (20e)$$

Calculations for Conventional Model Containing Fixed Number of Particles of Uniform but Variable Size. In calculations of evaporation and combustion it is usual to assume a model in which there is a constant number of particles of uniform size, the initial uniform size being set equal to the appropriate mean size of the initial spectrum. To see how this model compares with the true state of affairs, we shall determine, for this model, the expressions corresponding to those of Eqs. (20d) and (20e) for the spectrum of Eq. (20a). Let σ_μ and n_μ denote respectively the uniform but variable particle size and the constant number of particles for this model. Then by definition, $\sigma_{\mu, t=0}/\sigma^* = \sqrt{3}$ Furthermore, from Eq. (18),

$$\left(\frac{\sigma_\mu}{\sigma^*}\right)^2 = \left(\frac{\sigma_\mu}{\sigma^*}\right)^2_{t=0} - 2\theta = 3 - 2\theta \qquad (21a)$$

The fraction of the total volume of all particles which has disappeared at time t is given by

$$\chi_\mu = \left[1 - \frac{V}{V_{t=0}}\right]_\mu = 1 - \frac{\left(\mathcal{D}_\mu/\mathcal{D}^*\right)^3}{\left(\mathcal{D}_\mu/\mathcal{D}^*\right)^3_{t=0}} = 1 - \left(1 - \frac{2\theta}{3}\right)^{3/2} \tag{21b}$$

and the rate of evaporation or combustion (since $q \sim -\frac{dV}{dt}$) is, in proportion to the initial rate,

$$\left(g/g_o\right)_\mu = \left(1 - \frac{2\theta}{3}\right)^{1/2} \tag{21c}$$

Comparison of Conventional Model with Actual Cloud. Equations (20d), (20e), (21b), and (21c) are graphed in Fig. 5. Since by definition, the initial mass and initial rate of loss of mass are identical for the spectrum and model, Fig. 5 shows directly the inaccuracies in the model. At the beginning of the process, the two rates are equal, and consequently the slopes of the χ-curves are the same. As the process proceeds however, the rate calculated for the model is at first greater than the true rate; this occurs because, in the true spectrum, the instantaneous mean size does not change, whereas in the simple model, all particles grow smaller and consequently acquire a larger growth rate (in the absolute sense). After about 75% of the mass has disappeared (according to spectrum calculations), the rate calculated for the model becomes smaller than that for the spectrum; this happens because the excessively high rate at early times so reduces the particle diameter (and the surface area) of the model that ultimately a point must be reached where the rate becomes less than that of the spectrum. Notwithstanding the latter remark, at any instant the fraction disappeared is greater for the model than for the spectrum. Indeed, at $\theta = 1.5$, the particles of the model have entirely vanished, while in the spectrum they never entirely vanish.

To summarize, the model of evaporation or combustion in which the spectrum is represented by a constant number of particles of uniform but variable size yields too high an estimate of how much mass will have disappeared at any given time. The table below gives some significant comparisons.

	Model	Spectrum
Per cent disappearance for θ = .555	50%	42.6%
Per cent disappearance for θ = .905	75%	59.5%
θ for 50% disappeared	.555	.694
θ for 75% disappeared	.905	1.386
θ for 100% disappeared	1.5	∞

Proposed New Model of Particle Cloud. It is evident that the conventional model incorporates an unwarranted optimism concerning the time required for the process to occur. A more accurate model is suggested by the fact that the instantaneous $\overline{D}_{3/1}$ corresponding to Eq. (20b) is constant. That is, the actual spectrum may be replaced by a cloud of particles of variable number but of uniform and constant size, with the number of such particles decreasing at a rate proportional to the rate of evaporation or combustion. Such a model, for the initial spectrum of Eq. (20a) would give rates identical with those of the true spectrum. At first it may seem peculiar to imagine particles vanishing one by one, without diminution of diameter; but when it is recalled that the model is in any case a fiction to enable easy calculations, and that the number of particles in the model cloud has no physical association with the number in the true cloud, the seeming strangeness disappears.

It cannot be claimed that the model suggested above is identically correct for other particle-size distributions, or for cases where R is not inversely proportional to $\mathcal{D}$. However, since the spectral curve of

Eq. (20a) is at least typical in shape, and since R generally varies with $\mathcal{D}$ to a power lying between 0 and -1, it seems safe to say that the conventional model incorporating a constant number of drops is generally over-optimistic in its estimate of rates, and that the new model proposed is often more realistic.

Results for Nukiyama-Tanasawa Distribution

To investigate the two points mentioned in the preceding paragraph, we now examine the Nukiyama-Tanasawa distribution, accepted as best representing that of liquid sprays atomized by air jets [6]. The distribution equation at $t = 0$ is

$$\mathcal{D}^* G = 4 n_0 \left(\mathcal{D}/\mathcal{D}^* \right)^2 e^{-2 (\mathcal{D}/\mathcal{D}^*)} \quad \text{or} \quad G = 4 \mathbb{D}^2 e^{-2 \mathbb{D}} \qquad (22a)$$

where the constants are so chosen that n_0 is the total number of droplets and G is a maximum when $\mathbb{D} = 1$, all at $t = 0$.

Results for $R = - K/\mathcal{D}$. Referring to Eq. (19b), the solution for the size distribution at any time t is given by

$$G = 4 \mathbb{D} \left(\mathbb{D}^2 + 2 \theta \right)^{1/2} e^{-2 (\mathbb{D}^2 + 2 \theta)^{1/2}} \qquad (22b)$$

Inasmuch as the terms in θ cannot be brought out in a single factor, it follows that the shape of the spectrum changes with time. A few of the spectral curves are plotted in Fig. 6a, which shows that the value of $\mathbb{D}$ for which G is a maximum increases slightly with time.

Evaluation of Eqs. (2) and (4c) for the size distribution of Eq. (22) is somewhat lengthy, but the results can be expressed in terms of the second, fourth and sixth derivatives of the modified Bessel function of

the second kind of order zero. The results of these calculations are
plotted in Fig. 6b, which shows that the appropriate mean diameter for
this distribution actually increases with time. This result is perhaps
not too surprising after having seen the curves in Fig. 6a. This in-
crease in mean diameter, however, is not very large (about 17% when the
cloud is 75% evaporated).

Fig. 6c shows the fraction of mass disappeared vs. time for the
spectrum as well as the corresponding curves for the conventional model
with a constant number of drops of uniform but variable size, and for the
new model of constant diameter drops but varying number. The curves
illustrate clearly the error in using the conventional model for evapora-
ting drops. The proposed new model does not give results that coincide
exactly with the spectrum calculations; they are in error in the same
direction as the conventional model. This is expected because $\overline{D}$ actually
increases somewhat during evaporation and therefore the rate of change
of mass is less for the spectrum. However, the fraction evaporated
predicted by the proposed model is better than the conventional model
both qualitatively and quantitatively.

Results for R = Constant. The results shown in Fig. 6 are for the
case $R \sim \mathscr{D}^{-1}$. To see what effect the nature of the growth-rate law
has on the results, Fig. 7 shows curves analogous to those of Fig. 6,
for the same initial spectrum, but with R = constant = K_1. The two
cases comprise the two extreme forms of the growth-rate law, for in the
general relationship $R \sim \mathscr{D}^{-\gamma}$, γ lies between zero (radiant heat trans-
fer controlling), and unity (molecular conduction and diffusion controlling).
Note that the appropriate mean diameter for $\gamma = 0$ is $\overline{D}_{3/2}$.

Figure 7a shows the Nukiyama Tanasawa spectrum as a whole moving horizontally towards the origin. This result was previously derived for any spectrum if the growth rate is constant and uniform. In Fig. 7b, the mean diameter $\overline{D}_{3/2}$ is shown as a function of $\theta_1 \equiv K_1 t / D^*$. In this case the mean diameter decreases as the drops evaporate. Finally, in comparing the fraction evaporated curves in Fig. 7c, we see that once more the conventional model is over-optimistic about evaporation rates. The proposed model is also in error, but the magnitude of the error is smaller, and in the opposite direction. The proposed model again matches the spectrum result qualitatively over the entire range.

Spectrum with Finite Diameter Range

In the preceding examples the diameter range for each spectrum ranged from zero to infinity. As a consequence, an infinite amount of time is required to evaporate all the droplets. Since the upper diameter limit is finite in any actual case, we now investigate this effect using the molecular growth-rate law of Eq. (16) and the initial droplet spectrum of Eq. (20a). Thus at $t = 0$ we have

$$
G = \begin{cases} D\, e^{-D^2/2} & 0 \leq D \leq D_{max_{t=0}} \\ 0 & D > D_{max_{t=0}} \end{cases}
$$

The correct mean diameter is given by

$$
\overline{D}^2 = \frac{\int_0^{D_{max}} D^4 e^{-D^2/2}\, dD}{\int_0^{D_{max}} D^2 e^{-D^2/2}\, dD}
$$

where D_{max} is not constant but can be expressed as

$$
D_{max}^2 = D_{max_{t=0}}^2 - 2\theta
$$

The results of these calculations as well as the results of using the conventional model are shown in Fig. 8.

Choice of Initial Spectrum

The initial particle distribution G for any case is usually obtained by fitting a curve to experimental data. This procedure suggests trying to fit a curve of certain algebraic form such that later calculations for $\overline{\mathcal{D}}$ yield the result $\overline{\mathcal{D}}$ = constant. This objective can be achieved for the case of $f_i(\mathcal{D}) \sim - \mathcal{D}^{-\gamma}$ if we fit the data to the form

$$G_{t=0} = a \, \mathcal{D}^{\gamma} e^{-b \, \mathcal{D}^{\gamma+1}}$$

where a and b are arbitrary constants. Equation (20a) is a specific example of this for the case of $\gamma = 1$.

Conclusions of Droplet Evaporation Study.

The general results obtained from study of Fig. 6, 7 and 8 are that:

i) The conventional model is optimistic about evaporation rates of the spectrum.

ii) The proposed model is more accurate in predicting evaporation rates, at least for $\mathbb{D} \, e^{-\mathbb{D}^2}$ and $\mathbb{D}^2 e^{-\mathbb{D}}$ spectra, for the range of growth rates from radiation controlling ($\gamma = 0$) to molecular phenomena controlling ($\gamma = 1$) the process.

iii) If only a small part of the mass of the droplet spectrum is to be evaporated, it is immaterial which model is used because the diameter variation is so small ($\mathcal{D} \sim$ cube root of mass remaining).

iv) For evaporation of the last remnants of a droplet spectrum with $\mathbb{D}_{max} = \infty$, the results of the conventional model differ radically from those of the spectrum and of the proposed model. The latter two

indicate that 100% evaporation is reached only as t approaches ∞ .
If, however, D_{max} is finite, the spectrum analysis shows 100% evapora-
tion is reached in finite time while the proposed model still requires
infinite time. Except for this question of time for 100% evaporation,
the difference between the results of the proposed model and of the
spectrum analysis is small if $D_{max_{t=0}} > 3$. The physical reason for this
behavior is that very little of the mass in the droplet spectrum we
are considering is in the range from $D = 3$ to $D = \infty$. If the ques-
tion of time for 100% evaporation is of importance, the answer can
easily be obtained by considering just the evaporation of the largest
drop.

8. EVAPORATION OR COMBUSTION OF A STATIONARY CLOUD IN A "SMALL" MEDIUM

Form of Growth-Rate Law

When the growth rate depends on time, as for example when evapora-
tion or combustion modifies the temperature and other properties of
the medium, it is not valid to let $f_2(t) = 1$ in Eq. (13a).

In this event, if we retain the growth-rate law $R = f_1(D) f_2(t)$,
with $f_1(D) \sim D^{-1}$, Eq. (14) shows that all the results presented in Eqs.
(19), (20), and (22) and in Figs. 5 and 6 remain valid, provided only
that wherever t appears it must be replaced by $\int_0^t f_2(t)\, dt$.

Case of Evaporation in a Gaseous Medium

The details of completing the solution will vary from case to case,
and may best be illustrated by a specific example. Consider evapora-
tion into a cloud of gas, with the gas temperature falling by virtue
of the latent heat absorption. Then an energy balance for a droplet
(neglecting sensible heat effects), equating the rate of heat transfer

to the rate of latent heat change, yields

$$h \, \pi \, \mathcal{D}^2 \tau = - \rho \mathcal{L} \frac{\pi}{2} \, \mathcal{D}^2 \, (D\mathcal{D}/Dt)$$

whence, with the postulate of molecular heat conduction ($h\mathcal{D}/\lambda = 2$) we get

$$R = D\mathcal{D}/Dt = - 4 \lambda \tau / \rho \mathcal{L} \mathcal{D} \qquad (23a)$$

Neglecting the variation of λ with time, we may express this as

$$R = f_1 (\mathcal{D}) f_2(t) \quad \text{where} \quad f_1 \equiv - 4 \lambda \tau_0 / \rho \mathcal{L} \mathcal{D} \text{ and } f_2 \equiv \frac{\tau}{\tau_0}$$

An energy balance between the entire droplet cloud and the entire gas mass, equating the total latent heat supplied to the decrease of enthalpy of the gas, gives

$$m_g \, c_p \, (T_{g_0} - T_g) = \rho \mathcal{L} V_0 \left(1 - \frac{V}{V_0}\right) \qquad (23b)$$

Moreover, we have, by definition,

$$\tau \equiv T_g - T = (T_{g_0} - T) - (T_{g_0} - T_g) = \tau_0 - (T_{g_0} - T_g)$$

and thus we obtain

$$\tau = \tau_0 \left[1 - E \varkappa \right] \qquad (23c)$$

where

$$E \equiv \rho \mathcal{L} V_0 / m_g c_p \tau_0$$

Now, for each value of $\int_0^t f_2(t) \, dt = \int_0^t (\tau / \tau_0) \, dt$, the value of $\varkappa$ is known either from an analytical solution [e.g., Eq. (20d)] or, if necessary, from a numerical integration, keeping in mind that the value of Θ appearing in, say, Eq. (20d) is now to be interpreted as $(k/\mathcal{D}^{*2}) \int_0^t (\tau / \tau_0) \, dt$. Moreover, Eq. (23c) gives a relationship between τ / τ_0 and $\varkappa$, assuming of course that the constants τ_0, $\mathcal{L}$, ρ, c_p and V_0 / m_g are all known to begin with. Consequently, we have in hand a relationship between τ / τ_0 and $\int (\tau / \tau_0) \, dt$. Then, from the expression

$$\tau = \frac{d}{dt} \int_0^t \tau \, dt = \frac{d\tau}{dt} \frac{d}{d\tau} \int_0^t \tau \, dt$$

we may separate variables and integrate between $t = 0$, $\tau = \tau_0$, and $t = t$, $\tau = \tau$, thus getting the identity

$$t \equiv \int_1^{\tau/\tau_0} \frac{\frac{d}{d(\tau/\tau_0)} \int_0^t (\tau/\tau_0)\, dt}{(\tau/\tau_0)} \, d(\tau/\tau_0) \tag{23d}$$

Since the differentiation and integration required by the right-hand side of this equation may be performed analytically or graphically with the help of the aforementioned relationship between τ/τ_0 and $\int_0^t (\tau/\tau_0)\, dt$ we may solve for t as a function of τ/τ_0, either analytically or graphically. Then, since x is connected with τ/τ_0 through Eq. (23c), we have our solution finally in the desired form of x versus t.

Specific Example. For example, with the rather simple algebraic relationship of Eq. (20d), all the operations may be performed analytically, and yield finally, for the relationship between fraction evaporated and time,

$$t = \frac{\rho \mathcal{L} \mathcal{D}^{*2}}{4\lambda \tau_0 (E-1)} \ln \frac{1-x}{1-Ex} \tag{24}$$

With other spectra the operations indicated above may have to be carried out numerically or graphically.

9. STEADY STATE EVAPORATION OR COMBUSTION IN DUCTS

Specification of Model

Let us now consider the case of evaporation or combustion in steady flow in a duct, such that, at every station, all properties are constant, i.e., $\partial/\partial t = 0$.

Furthermore, let us suppose that the particles are very small, and therefore that all have virtually the same speed as the medium at all points, i.e., $u = u(s)$ only. For the same reasons as given in the preceding

section, let the growth rate be represented as $R = f_1(\mathcal{D}) f_2(s)$,

General Solution

With the foregoing assumptions, the application of Eqs. (9) yields, for the general solution to the problem,

$$G = e^{-N} H(J + S) \tag{25a}$$

where

$$J \equiv \int (1/f_1) d\mathcal{D}; \quad S \equiv \int (f_2/\mu) ds; \quad N \equiv \int \frac{\frac{\partial \mu}{\partial s} + \frac{\mu}{A} \frac{dA}{ds} + f_2 f_1'}{f_1 f_2} d\mathcal{D} \tag{25b}$$

and H is an arbitrary function of the argument $J + S$. To evaluate the N integral, it is necessary to express s as a function of $\mathcal{D}$ through the equation $J + S = J_{s=0}$, where $J_{s=0}$ is a constant of integration, equal to the value of J at $s = 0$

The actual treatment for particular cases is similar to that of the preceding section, where $\mathcal{D}$ and t were the independent variables, except that now $\mathcal{D}$ and s are the variables.

Special Case for which μA = Constant

When the density of the medium is not appreciably altered by the evaporation or combustion, and when the particles follow closely the motion of the medium, continuity requires that the product μA be nearly constant. Assuming that it is in fact constant, the integral N becomes

$$N = \ln f_1$$

And the solution takes the form

$$G f_1 = H(J + S) \tag{26}$$

which bears a striking similarity to Eq. (14).

General Solution for Mass Fraction Evaporated

The general solution for the spectral distribution as a function of distance, given by Eq. (25), may be very difficult to carry through. However,

should it be of interest only to discover how the fraction evaporated
or burned varies with time, simple physical reasoning shows that the
results of the preceding section (embodied, for example, in Eq. (20d)
and Fig. 6) are immediately applicable. If, as is the case with the
assumptions employed here, each portion of the particle cloud is assoc-
iated with a given mass of the medium, an observer moving with a given
portion of cloud and medium would see the same events that would be
seen if the cloud and medium were stationary. The only formula needed
is that relating the distance moved by the observer to the time elapsed,
and this is easily furnished by $s = \int u \, dt$. With this transformation,
all the results of the preceding section connecting fraction disappeared
with time may be used in the present section for connecting fraction
disappeared with distance.

10. STEADY FLOW, WITH NO CHANGE IN SIZE

Model of Process

Suppose that a cloud of particles travels in steady flow down a
duct, with no interactions which might cause any particle to change in
size. Then the concentration, G , might change because of (i) a change
in cross-sectional area of the duct, or (ii) a change in the speed, u ,
of the particles of size σ in question.

General Solution

With the foregoing assumptions, $\partial/\partial t = 0$ and $R = 0$. Accordingly
Eqs. (9b) are

$$\frac{ds}{u} = \frac{d\sigma}{0} = \frac{-d G}{(G/A)\frac{\partial}{\partial s}(u A)}$$

The integrals of these equations are

$$\sigma = c_1$$

and

$$G A u = c_2$$

and therefore the solution is

$$GA\mu = f(\sigma)$$

where $f(\sigma)$ signifies some function of σ .

Physical Meaning of Solution. This solution has a simple physical
meaning. Since G is the number concentration per unit volume, $GA\mu$ is
the number of particles in the size range from σ to $\sigma+d\sigma$, crossing
each section per unit time. Consequently, the solution states that,
for any particular size group, the number flux is the same at all cross-
sections. It would have been possible, therefore, to write down the
solution above simply by noting that the number of particles in each
size range is conserved.

We note from the solution that the number concentration per unit
volume, G , tends to decrease if either A or μ increases.

Example of Motion in Incompressible Medium, with Slow Changes in Cross-Section

Suppose that the particles are travelling in an incompressible medium,
and that the changes of cross-section are so moderate that all particles
follow exactly the motion of the medium. The the product μA is constant,
and it follows that the concentration curve, $G(\sigma)$, is the same at all
cross-sections.

Example of Motion in Incompressible Medium, with Rapid Contraction of
Cross-Section

By way of contrast, let the medium, assumed as incompressible,
accelerate through an abrupt decrease in cross-section, as in Fig. 9a.
For reasons of particle inertia, the smaller particles will accelerate
from 1 to 2 more rapidly than the larger ones. Ultimately, however,
all sizes of particles will arrive at the same speed at section 4.

Furthermore, the reasoning given previously requires the spectrum at 4
to be the same as at 1.   These arguments show that the spectral curves
for sections 1, 2, 3, and 4 will change in the general way indicated
by Fig. 9b.

11.   FREE-FALLING AND EVAPORATING DROPLET CLOUD

Model and Solution of Process

Imagine a cloud of falling and evaporating drops. Assuming a steady
state, we may set $\partial/\partial t = 0$.   Moreover, if each drop is very nearly at
its own terminal velocity at all times, we may suppose μ to be a function
of $\mathcal{D}$ only, and thus $\partial\mu/\partial s = 0$.   If, as well, the surrounding medium is
very large, we may suppose R to depend on $\mathcal{D}$ alone.   Then, noting that
$\partial A/\partial s = 0$, Eqs. (9b) become

$$\frac{ds}{\mu} = \frac{d\mathcal{D}}{R} = -\frac{dG}{G\,(\partial R/\partial\mathcal{D})}$$

The integral of the $ds,\, d\mathcal{D}$ equation is

$$s - S = c_1 \qquad \text{where} \quad S(\mathcal{D}) = \int \frac{\mu(\mathcal{D})}{R(\mathcal{D})}\, d\mathcal{D}$$

and the integral of the $d\mathcal{D},\, dG$ equation is

$$GR = c_2$$

Consequently the general solution is

$$GR = H(s - S)$$

in which the arbitrary function H is found from a knowledge of the
function GR at the location $s = 0$.

Example for Very Small Droplets (Low Reynolds Number)

To illustrate, suppose that the droplets are so small that the evap-
oration occurs by molecular diffusion and the terminal velocity is set by
Stokes' law of drag. These lead to the laws

$$R = -K/\mathcal{D} \; ; \quad \mu = K_2 \mathcal{D}^2$$

where K and K_2 are dimensional constants depending on the properties of the droplets and the medium. The function S may now be evaluated as

$$S \equiv \int \frac{K_2 \sigma^2}{-K/\sigma} d\sigma = - \frac{K_2}{4K} \sigma^4$$

and the general solution showing the spectral distribution as a function of S is seen to be

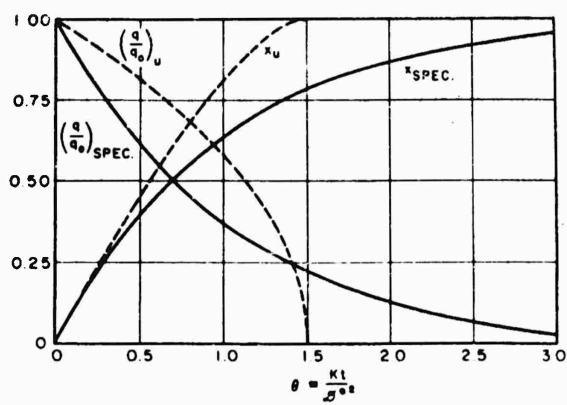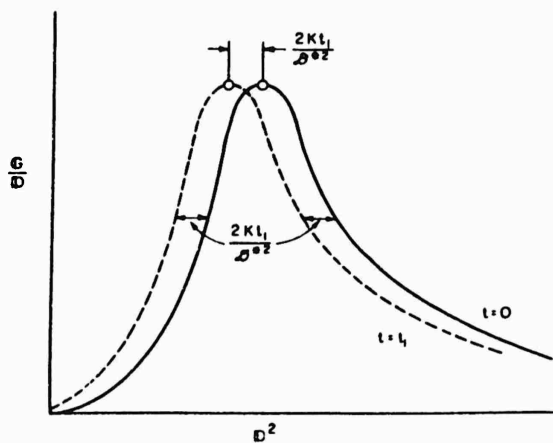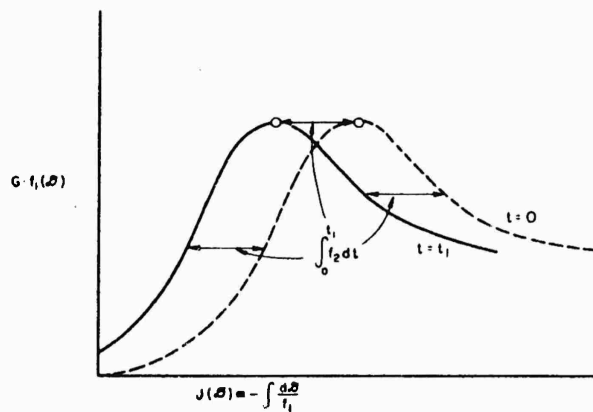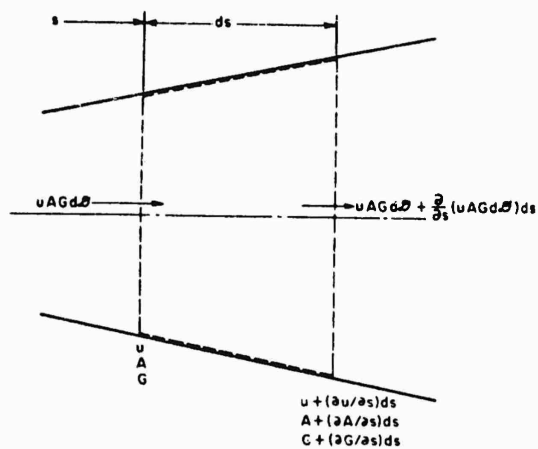$$G = \sigma H\left(s + \frac{K_2}{4K} \sigma^4\right)$$

Therefore, with a knowledge of how G/σ depends on σ^4 at $s = 0$, it is a simple matter to find $G(\sigma)$ at any other value of s.

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13. CAPTIONS TO FIGURES

Fig. 1. (a) Spectrum of particle sizes, illustrating nomenclature.

(b) Illustrates growth flux formula. Shaded zone shows number of droplets which, during the time δt , have grown from sizes smaller than σ to sizes larger than σ .

Fig. 2. Control space for formulation of differential equation.

Fig. 3. Graphical interpretation of spectral history for stationary cloud with the growth-rate law $R = f_1(\sigma) f_2(t)$

Fig. 4. Graphical solution for stationary cloud evaporating or burning in large gaseous medium, with $R = -K/\sigma$

Fig. 5. Comparison of fraction disappeared and rate of disappearance, both vs. time, for initial spectrum of Eq. (20a) (solid curves) and for simple model comprising a constant number of drops of uniform but variable size (dashed curves).

Fig. 6. Results for Nukiyama-Tanasawa spectrum of Eq. (22a), with $R = -K/\sigma$

(a) Spectral distributions at various times.

(b) Variation of equivalent mean diameter with fraction evaporated or burned.

(c) Fraction evaporated or burned vs. time for the spectrum, the conventional model, and the new model.

Fig. 7. Results for Nukiyama-Tanasawa spectrum of Eq. (22a), with constant value of R .

(a) Spectral distributions at various times.

(b) Variation of equivalent mean diameter with fraction evaporated or burned.

(c) Fraction evaporated or burned vs. time for the spectrum, the conventional model, and the new model.

Fig. 8. Variation of mean diameter with fraction evaporated or burned for spectrum of Eq. (20a) with an upper diameter limit D_{max}.

Fig. 9. Change of spectral distribution when particles carried by incompressible fluid pass through contraction in duct.
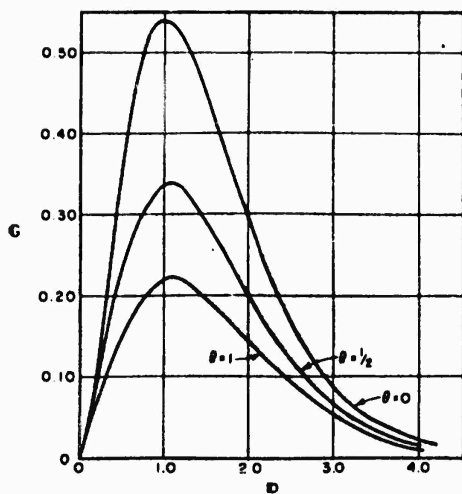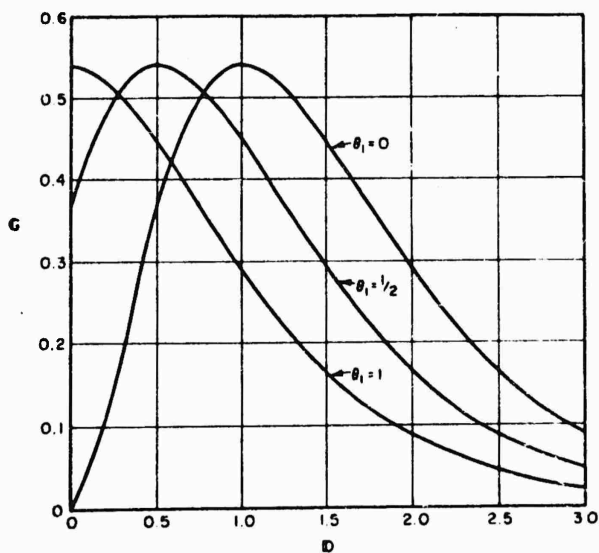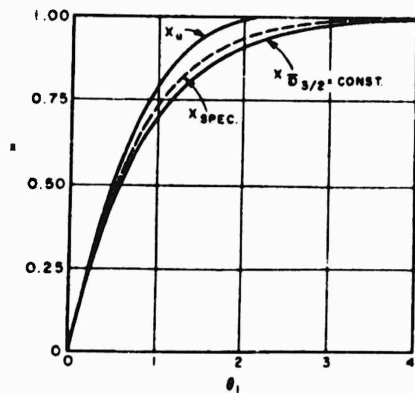
$G \equiv \dfrac{dn}{d\mathcal{B}}$
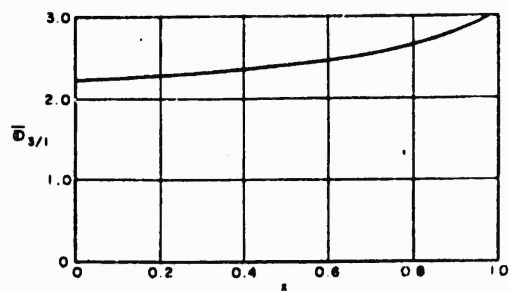
or

$\left[G \equiv \dfrac{d\left(\frac{n}{n_o}\right)}{d\left(\frac{\mathcal{B}}{\mathcal{B}^o}\right)} \right]$

Area = dn
= G·d$\mathcal{B}$

$\mathcal{B}$ or $\left[D \equiv \dfrac{\mathcal{B}}{\mathcal{B}^o} \right]$

FIG. 1a

No. of Drops = GRdt

FIG. 1b

$u A G d\mathcal{B}$ $u A G d\mathcal{B} + \dfrac{\partial}{\partial s}(u A G d\mathcal{B})ds$

u
A
G

$u + (\partial u/\partial s)ds$
$A + (\partial A/\partial s)ds$
$G + (\partial G/\partial s)ds$

FIG. 2

$G \cdot f_1(\mathcal{B})$

$\int_0^{t_1} f_2 dt$

$t = 0$

$t = t_1$

$J(\mathcal{B}) = -\int \dfrac{d\mathcal{B}}{f_1}$

FIG. 3

$\dfrac{G}{D}$

$\dfrac{2Kt_1}{\mathcal{B}^{o2}}$

$\dfrac{2Kt_1}{\mathcal{B}^{o2}}$

$t = 0$

$t = t_1$

D^2

FIG 4

$\left(\dfrac{q}{q_o}\right)_u$ x_u $x_{SPEC.}$

$\left(\dfrac{q}{q_o}\right)_{SPEC.}$

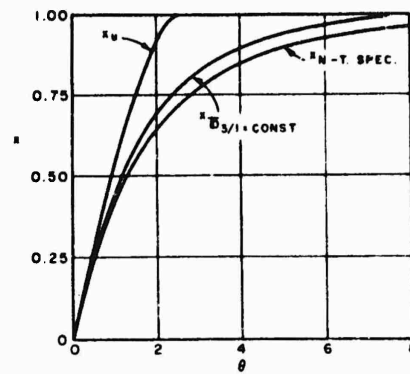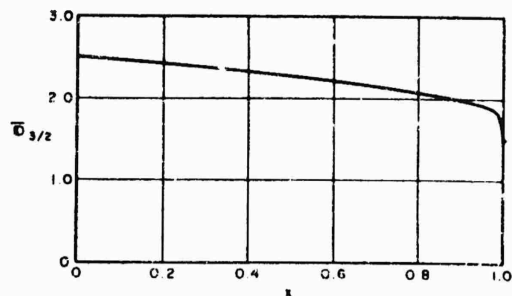$\theta = \dfrac{Kt}{\mathcal{B}^{o2}}$

FIG. 5

FIG. 6a

FIG. 6b

FIG. 6c

FIG. 7a

FIG. 7b

FIG. 7c

FIG. 8

FIG. 9a

FIG. 9b